THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, April 29, 1899. Twenty-eight persons were in attendance, including the following twenty-four members of the Society:

Professor Maxime Bôcher, Dr. A. S. Chessin, Professor F. N. Cole, Dr. W. S. Dennett, Professor T. S. Fiske, Mr. G. B. Germann, Dr. A. A. Himowich, Dr. J. I. Hutchinson, Professor Harold Jacoby, Mr. C. J. Keyser, Professor Pomeroy Ladue, Dr. Emory McClintock, Mr. James Maclay, Dr. D. A. Murray, Professor W. F. Osgood, Mr. J. C. Pfister, Professor James Pierpont, Professor J. K. Rees, Professor C. A. Scott, Dr. W. M. Strong, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Miss E. C. Williams, and Professor R. S. Woodward.

The President of the Society, Professor R. S. Woodward, occupied the chair during the morning and afternoon sessions. The Council announced the election of the following persons to membership in the Society: Professor Rosser D. Bohannan, Ohio State University, Columbus, Ohio; Professor Florian Cajori, Colorado College, Colorado Springs, Colo.; Mr. John S. Morris, Central High School, Philadelphia, Pa.; Dr. John V. E. Westfall, Cornell University, Ithaca, N. Y. Eleven applications for membership were reported. Professor Thomas F. Holgate was appointed Acting Secretary for the Summer Meeting.

The Council announced through the President that, the requisite financial guarantees having already been secured, it was prepared to recommend that the publication of the *Transactions* of the Society be begun in January, 1900. The By-Laws were then amended, in accordance with the recommendation of the Council, so as to define the status of the *Transactions* and to provide for the creation of its editorial board. This board consists of three editors appointed by the Council. As originally organized, it is composed of Professor Eliakim Hastings Moore, to serve until February, 1904; Professor Ernest W. Brown, to serve until February, 1903; and Professor Thomas Scott Fiske, to serve until February, 1902.

The following papers were presented at the meeting:

(1) Dr. J. I. HUTCHINSON: "The asymptotic lines of the Kummer surface."

(2) Dr. L. E. DICKSON: "The known finite simple groups."

(3) Mr. E. B. WILSON: "Note on the functions satisfying the functional relation $\varphi(x)\varphi(y) = \varphi(x+y)$."

(4) Dr. A. S. CHESSIN: "On the differential equations of dynamics."

(5) Professor CHARLOTTE ANGAS SCOTT: "A proof of Noether's fundamental theorem."

(6) Dr. G. P. STARKWEATHER: "Non-quaternion systems containing no skew units."

(7) Professor É. GOURSAT: "Sur la définition générale des fonctions analytiques d'aprés Cauchy."

(8) Professor F. MORLEY: The value of

$$\int_0^{\pi/2} (\log 2\cos\varphi)^n \varphi^n d\varphi."$$

(9) Professor E. W. BROWN: "An elementary illustration of the connection between the current and the height of the water in a tidal estuary."

(10) Dr. W. M. STRONG: "The determination of non-quaternion number systems in six units."

(11) Professor E. O. LOVETT: "Curves of multiple curvature."

(12) Professor JAMES PIERPONT: "Elliptic functions."

(13) Mr. C. J. KEYSER: "On a definitive property of the covariant."

In the absence of the authors the papers of Professor Morley, Professor Lovett, and Mr. Keyser were read by title; Dr. Dickson's paper was read by the Secretary, Mr. Wilson's and Professor Goursat's by Professor Osgood, and Professor Brown's by Professor Woodward. The papers of Mr. Wilson and Professor Goursat were offered to the Society through Professor Osgood, Dr. Starkweather's through Professor Pierpont. The papers of Dr. Hutchinson, Dr. Dickson, Professor Pierpont, and Mr. Keyser will appear in later numbers of the BULLETIN. Abstracts of the other papers are given below.

Mr. Wilson showed that if a function exists which is defined for all values of the argument, is single valued, satisfies the functional relation $\varphi(x)\varphi(y) = \varphi(x+y)$, and has a discontinuity at a single point, then, in the neighborhood of any value x_0 of the argument whatsoever, the function takes on values arbitrarily near to any preassigned positive value y_0. This paper will be published in the *Annals of Mathematics*.

Professor Chessin's paper is in abstract as follows: When the vincula (liaisons) of a moving system contain t explicitly in their equations, we may integrate the differential equations of motion as follows. Let these equations be given in canonic form

$$(1) \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}; \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}.$$

Then we may put

$$H = H_0 + \Omega$$

where H_0 denotes the Hamiltonian function H with the assumption that $t = \text{const.}$ in the equations of the vincula. We therefore first integrate the system

$$(2) \qquad \frac{dp}{dt} = -\frac{\partial H_0}{\partial q}; \qquad \frac{dq}{dt} = \frac{\partial H_0}{\partial q},$$

and then, as in the theory of perturbations, pass to the integrals of (1) by integrating the canonic system

$$(3) \qquad \frac{d\alpha}{dt} = \frac{\partial \Omega}{\partial \beta}; \qquad \frac{d\beta}{dt} = -\frac{\partial \Omega}{\partial \alpha},$$

the α, β being the constants of integration of system (2).

If it is possible to eliminate t from the equations of the vincula by introducing a moving system of coördinates ($O.\Xi YZ$) instead of the fixed system (XYZ), then Ω may be expressed as follows:

$$\Omega = -L - \left(\frac{dN}{dt} - \frac{\partial N}{\partial t}\right) - \tfrac{1}{2}\Sigma\Sigma\phi_{\lambda\mu}\frac{\partial \varphi_\lambda}{\partial t}\frac{\partial \varphi_\mu}{\partial t}$$

where $\frac{\partial}{\partial t}$ means partial differentiation with regard to t, while the other letters denote respectively:

$$L = \omega M_0 \cos(\omega M_0),$$

ω being the angular velocity of rotation for the system ($O.\Xi YZ$) and M_0 the principal moment about O of the momentum of relative motion of the given material system (*i. e.*, relatively to the system ΞYZ);

$$N = m w_0 \rho_c \cos(w_0, \rho_c),$$

m being the total mass of the given system, w_0 the velocity

of the point O, ρ_c the radius vector of the center of inertia (from O);

$$\Phi_{\lambda\mu} = \frac{\partial lg\Phi}{\partial \varphi_{\lambda\mu}}$$

$$\Phi = |\, \varphi_{\lambda\mu} \,|$$

$$\varphi_{\lambda\mu} = \Sigma \frac{1}{m}\left(\frac{\partial \varphi_\lambda}{\partial x_i}\frac{\partial \varphi_\mu}{\partial x_i} + \frac{\partial \varphi_\lambda}{\partial y_i}\frac{\partial \varphi_\mu}{\partial y_i} + \frac{\partial \varphi_\lambda}{\partial z_i}\frac{\partial \varphi_\mu}{\partial z_i}\right);$$

and, finally,

$$\varphi_\lambda(x_1, y_1, z_1, \cdots, x_i, y_i, z_i, \cdots, t) = 0,$$

.....

being the equations of the vincula.

For a solid body a similar expression is obtained for Ω, the difference being only in the expressions $\varphi_{\lambda\mu}$.

The method here discussed is an application of the author's treatment of relative motion in general, as mentioned in two previous papers read before this Society (August 19 and October 29, 1898).

The proofs hitherto given of Noether's fundamental theorem are purely algebraic, befitting the algebraic use of the theorem. But it is also of primary importance in geometry, hence it seems worth while giving a proof that depends on geometrical ideas, not on elimination and convergent expansions. Professor Scott's paper presents such a proof. Let curves U, V intersect at points of multiplicity q_1r_1, q_2r_2, ... on the two curves, and consider a curve through these points, with multiplicity still to be determined. Denote any curve that has this multiplicity by Ω. It is shown that if every Ω of a given order N, *i. e.*, every Ω_N, is of the form $BU + AV$, then this holds also for the next lower order, *i. e.*, $\Omega_{N-1} \equiv B_1U + A_1V$, hence for every Ω of order $< N$. Now the conditions imposed by the points are certainly independent if N be sufficiently great; and in this case the expressibility of Ω in the form $BU + AV$ requires that the multiplicity at the intersections be $\geqq q + r - 1$. Thus Noether's theorem is proved for the case that is of most general applicability in geometrical investigations. This paper will appear in the *Mathematische Annalen*.

Dr. Starkweather discussed a special class of non-quaternion number systems. It can be assumed that the given system is irreducible, otherwise it falls into irreducible sys-

tems of the type considered. We can take for one unit the modulus, η. Scheffers* has shown that, the system being of order u, we can take the remaining units $u_1, u_2, \cdots, u_{n-1}$, such that u_iu_j and u_ju_i $(j \leqq i)$ are linear in $u_1, u_2, \cdots, u_{j-1}$, and that any number expressible in these u's, say v, satisfies the equation $v^\lambda = 0$ where $\lambda \leqq n$.

Let $\lambda = n - \delta$. δ is called the deficiency of the system. Then a number σ must exist such that $\sigma^{n-\delta-1} \neq 0$. The quantities $\sigma, \sigma^2, \cdots, \sigma^{n-\delta-1}$ are linearly independent, and can be used as units $w_1 = \sigma^{n-\delta-1}$, $w_2 = \sigma^{n-\delta-2}$, $\cdots$, $w_k = \sigma^{n-\delta-k}$, $w_{n-\delta-1} = \sigma$. The multiplication table for the w's follows at once. There must be δ more units aside from η. It is possible to choose for these certain of the u's, $u_{z_1}, u_{z_2}, \cdots, u_{z_\delta}$, where $z_1 < z_2 \cdots < z_\delta$, such that $u_1, \cdots, u_{z_1-1}$ are expressible in the w's, and $u_{z_a+1}, \cdots, u_{z_{a+1}-1}$ are expressible in the w's and $u_{z_1}, u_{z_2}, \cdots, u_{z_a}$. By replacing u_{z_a} $(a = 1, 2, \cdots, \delta)$ by

$$\tau_a = u_{z_a} + a_{a,2}w_2 + a_{a,3}w_3 + \cdots + a_{a,(n-\delta-1)}w_{n-\delta-1}$$

it is possible to have the table assume the following form: $\tau_a w_{n-\delta-k}$ and $w_{n-\delta-k}\tau_a$ are zero if $k > a$; contain only w_1 if $k = a$; and if $k < a$ they are linear functions of

$$w_1, \cdots, w_{a-k+1};\ \tau_1, \cdots, \tau_{a-k}.$$

$\tau_a\tau_\beta (\beta \leqq a)$ is a linear function of $w_1, \cdots, w_a$; $\tau_1, \cdots, \tau_{a-1}$. τ_δ occurs only in its column and row, in the product with η. If we remove this column and row we have a system in $n-1$ units. Since σ is not removed the equation satisfied by any number formed from the units $w_1, \cdots, w_{n-\delta-1}$; $\tau_1, \cdots, \tau_{\delta-1}$, is still $v^{n-\delta} = 0$ or $v^{(n-1)-(\delta-1)} = 0$. So our system in $n-1$ units is of deficiency $\delta - 1$. Hence every number system of the type considered in n units, of deficiency δ, can be obtained by taking each independent number system in $n-1$ units of deficiency $\delta - 1$ in the w, τ form and bordering it, as indicated, by a row and column τ_δ. There will be certain parameters introduced in this bordering which can be restricted by application of the associative law and by the fact that $\tau_\delta{}^{n-\delta} = 0$.

Professor Goursat's paper, which will be published in the *Transactions*, is in abstract as follows: The proof of Cauchy's integral theorem

$$\int_C f(z)dz = 0$$

* *Math. Annalen*, vol. 39.

which Professor Goursat obtained in 1883 (cf. *Acta Mathematica*, vol. 4 (1884), pp. 197–200; Harkness and Morley, Theory of Functions, p. 164) depends on the possibility of dividing up the region A, along whose contour C the integral $\int_C f(z)dz$ is to be extended, into small squares, or pieces of squares, in such a manner that the sum of the integrals $\int f(z)dz$ taken along each of these little contours becomes less in absolute value than an arbitrarily small positive quantity. The value of this sum is precisely $\int_C f(z)dz$, and hence this integral vanishes. The proof turns on the fact that, ϵ being a positive quantity chosen at pleasure, the squares can be taken of such a size that, z_i being a point properly chosen within or on the boundary of the ith square and z being any point on the boundary of this square, the relation

$$|f(z) - f(z_i) - (z - z_i)f'(z_i)| \leqq |z - z_i|\,\epsilon$$

will be satisfied. In the present paper Professor Goursat shows that this relation can be established by the aid of the *sole assumption that $f(z)$ possesses at each point of the region in question a derivative, i. e.*, that

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

converges toward a finite limit at the point z_0, which is any point of the region. From Cauchy's integral theorem follows at once Cauchy's integral formula

$$f(x) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z - x},$$

and on these two theorems the whole theory of functions of a complex variable can be built up. Thus the question of whether in the definition of an analytic function it is necessary to require, in addition to the *existence* of a derivative, that the derivative shall be *continuous*—a question which up to the present time has remained open—is answered in the negative; the continuity of the derivative follows from its existence. The method employed in the proof is the familiar one of assuming that for a given square (or piece of a square) the relation is not satisfied, and then of subdividing the square. The assumption that a finite number of

repetitions of the process of subdivision will not yield squares for each of which the relation can be satisfied leads to the conclusion that there exists a point z_0 within or on the boundary of the original square such that for some contours in every neighborhood of z_0 the relation cannot be satisfied, and here is a contradiction.

The theory of the function $\Gamma(1+x)$ can be used to obtain the values of certain classes of definite integrals; these classes include some of those obtained in very artificial ways in the ordinary text-books. Professor Morley's note exhibits the process in the case of the integral

$$\int_0^{\pi/2} (\log 2\cos\varphi)^m \varphi^n d\varphi$$

where m is any positive integer, n any positive even integer.

Professor Brown's paper presented an interesting illustration of a special form of tidal motion, supplementing a passage in the author's review of Darwin's work on Tides in the May number of the BULLETIN.

In his article in the *Mathematische Annalen*, vol. 39, Scheffers has given general methods by which the non-quaternion number systems of ranks 2 and 5 in six units can be determined. Dr. Strong has determined the remaining systems in six units.

Professor Lovett's paper studies the theory of curves of multiple curvature by the methods of the intrinsic geometry as developed recently by Cesàro. A line is said to be of triple curvature when no five consecutive points of it lie in the same linear space of three dimensions. The right line determined by two consecutive points, the plane determined by three consecutive points, and the linear space of three dimensions S_3 determined by four consecutive points are called respectively tangent, osculating plane, and osculating space. At every point of the curve there are ∞^2 normals; these all lie in the same S_3, which is called the normal space. Among these ∞^2 normals one is found in the osculating plane; it is called the principal normal. ∞^1 of the normals are perpendicular to the osculating plane; they are called binormals because they are perpendicular to two consecutive tangents. One only of the binormals is contained in the osculating space; it is called the principal binormal. The

only binormal perpendicular to the osculating space is called the trinormal because perpendicular to three consecutive tangents. The tangent, trinormal, principal binormal, and principal normal at the same point are the principal directions of the curve at this point. The tetrarectangular tetrahedroid whose edges are the principal directions is taken as the fundamental tetrahedroid movable with the point. The following fundamental formulæ are derived and employed to investigate a curve at any one of its points:

$$\frac{dx}{ds} = \frac{t}{\rho} - 1, \quad \frac{dy}{ds} = \frac{z}{R}, \quad \frac{dz}{ds} = \frac{t}{\tau} - \frac{y}{R}, \quad \frac{dt}{ds} = -\frac{x}{\rho} - \frac{z}{\tau};$$

they express the necessary and sufficient conditions that a point in space remain fixed when the fundamental tetrahedroid is given an infinitesimal displacement along the curve. The analogous formulæ for the invariance of a direction whose cosines are α, β, γ, ε are

$$\frac{d\alpha}{ds} = \frac{t}{\rho}, \quad \frac{d\beta}{ds} = \frac{\gamma}{R}, \quad \frac{d\gamma}{ds} = \frac{\varepsilon}{\tau} - \frac{\gamma}{R}, \quad \frac{dt}{ds} = -\frac{\alpha}{\rho} - \frac{\gamma}{\tau}.$$

In these formulæ ρ, γ, and R measure respectively the rapidity of the deviations of the curve from its tangent line, its osculating plane and its osculating space, respectively.

These relations are used to demonstrate that a triad of independent equations

$$f(s, \rho, \tau, R) = 0, \quad g(s, \rho, \gamma, R) = 0, \quad g(s, \rho, \gamma, R) = 0$$

determines completely and uniquely a curve of triple curvature. The results of the integration yield in turn elegant expressions for the curvatures and principal directions at any point of the curve. The paper concludes with an extension of the barycentric analysis to a study of pentahedroidal potential. A novel theorem that may be mentioned in this abstract is that a curve of multiple curvature cuts its osculating space of highest dimensions or lies wholly on one side of that space according as the number of dimensions of the space necessary to the existence of the curve is odd or even. The paper will appear in the last number of the current volume of the *Annals of Mathematics*.

F. N. COLE.

COLUMBIA UNIVERSITY.

SURFACES OF REVOLUTION IN THE THEORY OF LAMÉ'S PRODUCTS.

BY DR. F. H. SAFFORD.

(Read before the American Mathematical Society at the Meeting of February 25, 1899.)

THE present paper is a review of an article by Haentzschel* in which he criticises certain results obtained by Wangerin.† The problem treated by Wangerin is to obtain the most general orthogonal surfaces of revolution, such that, if Laplace's equation be written in coördinates corresponding to these surfaces, a solution may be obtained in the form of a Lamé's product with an extraneous factor, *i. e.*,

$$V = \lambda . R . R_1 . \theta . \tag{1}$$

R, R_1, θ are functions respectively of the parameters of the two families of surfaces and the meridian planes, while λ may contain all three parameters. Wangerin shows that λ is $\frac{1}{\sqrt{r}}$, where r is the distance from the axis of revolution to the intersection of the three surfaces. His principal result is that the meridian curves are of the fourth degree and are cyclic curves, while the surfaces are of the same degree. Haentzschel states that the most general surfaces are of the thirty-second degree, the meridian curves being of the sixteenth degree.

Both writers give the following equations:

$$\frac{F'(t+iu) \cdot F_1'(t-iu)}{[F(t+iu) - F_1(t-iu)]^2} = H(t) + H_1(u), \tag{2}$$

$$x + ir = F(t+iu), \quad x - ir = F_1(t-iu) \quad (r = \sqrt{y^2 + z^2}). \tag{3}$$

By (2) two conjugate imaginary functions F and F_1 are to be determined such that the first member shall be the sum of two functions, one of t alone, the other of u alone. From (3) the two families of meridian curves are to be obtained by elimination of t and u respectively.

After differentiating (2) successively with respect to t and u, the result is

* Reduction der Potentialgleichung, E. Haentzschel, Berlin, 1893.

† *Berliner Monat-berichte*, Feb., 1878.

$$(F-F_1)\,[(F-F_1)\,(F'''F_1'-F'F_1''')$$
$$+(F''F_1'-F'F_1'')\,(F'-F_1')-2F'F_1'\,(F''+F_1'')$$
$$-2\,(F'+F_1')\,(F'F_1''+F''F_1')]$$
$$-3\,(F'-F_1')\,[(F-F_1)\,(F''F_1'-F'F_1'')$$
$$-2F'F_1'\,(F'+F_1')]=0. \qquad (4)$$

This expression is next differentiated three times with respect to the argument of F and then the derivatives of F_1 are eliminated. From this result, after integration, comes the following equation *defining* F:

$$(F')^2=AF^4+4BF^3+6CF^2+4B'F+A'=R(F). \qquad (5)$$

Corresponding to (5),

$$(F_1')^2=\overline{A}F_1^4+4\overline{B}F_1^3+6\overline{C}F_1^2+4\overline{B}_1'F+\overline{A}'. \qquad (6)$$

The conjugate imaginary constants in (5) and (6) come from integration but must be taken *real*, since (4) is satisfied by F, F_1, and their derivatives from (5) and (6) when and only when the following condition holds:

$$A-\overline{A}=B-\overline{B}=C-\overline{C}=B'-\overline{B}'=A'-\overline{A}'=0. \qquad (7)$$

Haentzschel gives, as the general integral of (5)

$$F(t+iu)^*=z_0+$$
$$\{\sqrt{R(z_0)}\,\sqrt{4s^3-g_2s-g_3}+\tfrac{1}{2}R'(z_0)[s-\tfrac{1}{24}R''(z_0)] \qquad (8)$$
$$+\tfrac{1}{24}R(z_0)\,R'''(z_0)\}\div\{2[s-\tfrac{1}{24}R''(z_0)]^2-\tfrac{1}{2}AR(z_0)\},$$

$$g_2=AR(z_0)-\frac{R'(z_0)R'''(z_0)}{24}+\frac{R''^2(z_0)}{48}$$
$$=-4(\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1), \qquad (9)$$

$$g_3=\frac{AR(z_0)R''(z_0)}{12}+\frac{R'(z_0)R''(z_0)R'''(z_0)}{24^2}-\frac{R(z_0)R'''^2(z_0)}{24^2}$$
$$-\frac{AR'^2(z_0)}{16}-\frac{R'''^3(z_0)}{12^3}=4\varepsilon_1\varepsilon_2\varepsilon_3, \qquad (10)$$

$$s=\wp(t+iu),\quad \sigma=\wp(t-iu). \qquad (11)$$

* Enneper, Elliptische Functionen, Halle, 1890, pp. 27–30.

The constant z_0 is at present unrestricted. When z_0 is a root of the equation

$$R(z) = 0, \tag{12}$$

(8) reduces to

$$F(t+iu) = z_0 + \frac{\frac{1}{4}R'(z_0)}{s - \frac{1}{24}R'''(z_0)}. \tag{13}$$

This result is used by Haentzschel to obtain meridian curves of the fourth degree and surfaces of both the fourth and eighth degrees, depending upon the nature of the root of (12) which is used in (13). The writer* has discussed a linear transformation of (13) and obtained restrictions analogous to (7), giving results identical with Wangerin's, viz., surfaces of the fourth degree only.

In the general case, where (12) no longer holds, it is convenient to change (5) to a form similar to Legendre's normal form, by using

$$F = \frac{a\overline{\overline{F}} + \beta}{\gamma\overline{\overline{F}} + \delta}, \tag{14}$$

Hence
$$(\overline{\overline{F}}')^2 = \overline{A}_0[\underset{1}{\pm}\,\overline{\overline{F}}^4 \underset{2}{\pm} \overline{c}\overline{\overline{F}}^2 \underset{3}{\pm} 1] \tag{15}$$

The three independent double signs in (15) correspond to the eight possible cases. It may be shown possible to choose real coefficients in (14) which will make $\overline{A}_0$ positive and $\overline{c}$ real in (15), while (2) is unchanged in form. Next, assume

$$\overline{\overline{F}} = \overline{F} - z_0 = v = \overline{x} + i\overline{r} - z_0. \tag{16}$$

From (15) and (16)

$$(\overline{F}')^2 = \overline{A}_0\,[\underset{1}{\pm}\,(\overline{F}-z_0)^4 \underset{2}{\pm} \overline{c}\,(\overline{F}-z_0)^2 \underset{3}{\pm} 1] = \overline{R}\,(\overline{F}). \tag{17}$$

Hence
$$\overline{R}\,(z_0) = \underset{3}{\pm}\,\overline{A}_0, \quad \overline{R}'\,(z_0) = 0, \tag{18}$$

$$\frac{\overline{R}''\,(z_0)}{2\,!} = \underset{2}{\pm}\,\overline{c}\overline{A}_0, \quad \frac{\overline{R}'''\,(z_0)}{3\,!} = 0, \quad \frac{\overline{R}^{\mathrm{v}}\,(z_0)}{4\,!} = \underset{1}{\pm}\,\overline{A}_0.$$

Reference to the deduction of (7) will show that z_0 must be real. Equations (17) (18) may be improved in form by the following changes:

$$\underset{3}{\pm}\,\overline{A}_0 = A_0, \quad (\underset{2}{\pm})\,(\underset{3}{\pm})\,\overline{c} = a, \quad (\underset{1}{\pm})\,(\underset{3}{\pm}) = \pm. \tag{19}$$

* *Amer. Jour. of Math.*, vol. 21, no. 1, pp. 11–23.

Equation (8), which is the solution of (17), is next simplified by the use of (16) and (18) and then solved for s, giving

$$s = A_0 \frac{\left(1 + \frac{av^2}{6} + \sqrt{\pm v^4 + av^2 + 1}\right)}{2v^2} \qquad (20)$$

Similarly for σ, the conjugate of s,

$$\sigma = A_0 \frac{\left(1 + \frac{aw^2}{6} \sqrt{\pm w^4 + aw^2 + 1}\right)}{2w^2}. \qquad (21)$$

From (9) and (10), corresponding to the *upper signs* in (20) and (21),

$$g_2 = \frac{A_0^4}{2}(a^2+12), \quad g_3 = \frac{A_0^6 a}{6^3}(-a^2+36), \qquad (22)$$

$$e_1 = \frac{A_0}{12}(a+6), \quad e_2 = -\frac{A_0 a}{6}, \quad e_3 = \frac{A_0}{12}(a-6). \qquad (23)$$

The equations of the two families of meridian curves are to be obtained by eliminating t and u respectively from (11) and then substituting the values s and σ from (20) and (21) in those results. Haentzschel has performed the elimination and substitution but his final result is of the sixteenth degree and he was probably unaware that it is reducible to four curves of the fourth degree. From (11) Haentzschel has obtained on page 17

$$s^2\sigma^2 + \alpha + (s+\sigma)^2\beta + s\sigma(s+\sigma)\gamma + s\sigma\delta + (s+\sigma)\varepsilon = 0. \qquad (24)$$

The values of the coefficients are given below and are functions of the parameter ρ, equal respectively to $\wp(2iu)$ and $\wp(2t)$.

Let ρ be replaced by $A_0\varphi$, then by the aid of (22),

$$\left.\begin{aligned}
\alpha &= g_3\rho + \frac{g_2^2}{16} = A_0^4\left(\frac{a^4}{48^2} - \frac{\varphi a^3}{6^3} + \frac{a^2}{96} + \frac{\varphi a}{6} + \frac{1}{16}\right),\\
\beta &= A_0^2\varphi^2, \qquad \gamma = -2A_0\varphi,\\
\delta &= -4\left(\rho^2 - \frac{g_2}{8}\right) = A_0^2\left(\frac{a^2}{24} + \frac{1}{2} - 4\varphi^2\right),\\
\varepsilon &= \frac{g_2\rho}{2} + g_3 = A_0^3\left(-\frac{a^3}{6^3} + \frac{\varphi a^2}{24} + \frac{a}{6} + \frac{\varphi}{2}\right).
\end{aligned}\right\} \qquad (25)$$

From (20), (21), (24), (25) comes the following equation of the curves desired:

$$
\begin{aligned}
&v^4w^4\left(2+\frac{13a^2}{18}+\frac{8\varphi a}{3}+8\varphi^2\right)\\
&+(v^4w^2+v^2w^4)\left(3a+\frac{a^3}{36}-\frac{2\varphi a^2}{3}+4\varphi^2 a\right)\\
&+v^2w^2\left(2+\frac{35a^2}{18}-\frac{32\varphi a}{3}-8\varphi^2\right)\\
&+(v^4+w^4)\left(2+\frac{a^2}{18}-\frac{4\varphi a}{3}+8\varphi^2\right)\\
&+(v^2+w^2)\left(\frac{8a}{3}-8\varphi\right)+4\\
&+\sqrt{v_0}\sqrt{w_0}\left[v^2w^2\left(2+\frac{5a^2}{18}-\frac{8\varphi a}{3}-8\varphi^2\right)\right.\\
&\left.+(v^2+w^2)\left(\frac{2a}{3}-8\varphi\right)+4\right]\\
=&-\sqrt{v_0}\left[v^2w^4(2a)+v^2w^2\left(2+\frac{11a^2}{18}-\frac{20\varphi a}{3}-8\varphi^2\right)\right.\\
&+w^4\left(2+\frac{a^2}{18}-\frac{4\varphi a}{3}+8\varphi^2\right)+w^2\left(\frac{8a}{3}-8\varphi\right)\\
&\left.+v^2\left(\frac{2a}{3}-8\varphi\right)+4\right]\\
&-\sqrt{w_0}\left[v^4w^2(2a)+v^2w^2\left(2+\frac{11a^2}{18}-\frac{20\varphi a}{3}-8\varphi^2\right)\right.\\
&+v^4\left(2+\frac{a^2}{18}-\frac{4\varphi a}{3}+8\varphi^2\right)+v^2\left(\frac{8a}{3}-8\varphi\right)\\
&\left.+w^2\left(\frac{2a}{3}-8\varphi\right)+4\right], \qquad (26)
\end{aligned}
$$

$$v_0=v^4+av^2+1, \quad w_0=w^4+aw^2+1. \qquad (27)$$

From (26), by squaring both members, comes

$$A_1(v^4w^4+1)+B_1(v^4w^3+v^3w^4+v^3+w^3)+C_1(v^4+w^4)$$

$$+D_1v^2w^2=-\sqrt{v_0}\sqrt{w_0}\,[E_1(v^2w^2+1)+H_1(v^2+w^2)]. \quad (28)$$

The coefficients A_1, B_1, etc., are functions of φ and a of degree not exceeding the sixth and containing from seven to fourteen terms each. From (28) by squaring both members the rational equation of the curves is obtained, containing more than eleven thousand terms, if expanded completely. The coefficients are functions of a, A_1, B_1, etc., and may be expressed in more convenient form by the following substitutions,

$$b=\frac{2a}{3}+4\varphi,\quad c=1-\frac{a^2}{6^2}+\frac{2\varphi a}{3}-4\varphi^2, \quad (29)$$

$$k=(a-2)^2(a+2)^2, \quad (30)$$

Equation (17) would degenerate if k were to vanish. The equation obtained from (28) is, after division by k,

$$(v^8w^8+1)b^4+(v^8+w^8)c^4+4(v^8w^6+v^6w^8+v^2+w^2)b^3c$$

$$+4(v^8w^2+v^2w^8+v^6+w^6)bc^3+6(v^8w^4+v^4w^8+v^4+w^4)b^2c$$

$$+2v^4w^4(3b^4+3c^4+12b^2c^2+8a^2b^2c^2-8ab^3c-8abc^3)$$

$$+4(v^6w^6+v^2w^2)(c^2-b^2+2abc)b^2$$

$$+4(v^6w^4+v^4w^6+v^4w^2+v^2w^4)(4abc-b^2-c^2)bc$$

$$+4(v^6w^2+v^2w^6)(b^2-c^2+2abc)c^2=0. \quad (31)$$

Equation (31) may be factored, giving

$$\left[(v^2w^2+1)\left(4\varphi+\frac{2a}{3}\right)+(v^2+w^2)\left(1-\frac{a^2}{6^2}+\frac{2\varphi a}{3}-4\varphi^2\right)\right.$$

$$\left.+vw\left(2-\frac{5a^2}{18}+\frac{8\varphi a}{3}+8\varphi^2\right)\right]^2$$

$$\left[(v^2w^2+1)\left(4\varphi+\frac{2a}{3}\right)+(v^2+w^2)\left(1-\frac{a^2}{6^2}+\frac{2\varphi a}{3}-4\varphi^2\right)\right.$$

$$\left.-vw\left(2-\frac{5a^2}{18}+\frac{8\varphi a}{3}+8\varphi^2\right)\right]^2=0. \quad (32)$$

Finally from (32) after introducing ρ again in place of φ and making use of (23).

$$\left[\frac{(v+w)^2}{\rho-\varepsilon_2}+\frac{(vw-1)^2}{\rho-\varepsilon_3}-\frac{(vw+1)^2}{\rho-\varepsilon_1}\right]^2$$

$$\left[-\frac{(v-w)^2}{\rho-\varepsilon_2}+\frac{(vw-1)^2}{\rho-\varepsilon_1}-\frac{(vw+1)^2}{\rho-\varepsilon_3}\right]^2=0. \quad (33)$$

These are the familiar cyclic curves and need no further discussion.

If in (20) and (21) the lower signs had been taken the computation would have been identical. for the following changes lead to an equation of the form of (26)

$$\bar{a}=-ai, \quad \bar{v}^2=v^2i, \quad \bar{w}^2=w^2i, \quad \bar{\rho}=-\rho i, \text{ etc.}$$

The result is

$$\left[\frac{(v+w)^2i}{\rho-\varepsilon_2}+\frac{(vw-i)^2}{\rho-\varepsilon_3}-\frac{(vw+i)^2}{\rho-\varepsilon_1}\right]^2$$

$$\left[-\frac{(v-w)^2i}{\rho-\varepsilon_2}+\frac{(vw-i)^2}{\rho-\varepsilon_1}-\frac{(vw+i)^2}{\rho-\varepsilon_3}\right]^2=0. \quad (34)$$

Haentzschel notices the fact that (3) may be replaced by the following:

$$r+ix=F(t+iu), \quad r-ix=F_1(t-iu). \quad (35)$$

There will be changes in several equations, especially in (7), but, as Haentzschel admits, nothing new will result.

From the forms of the cyclic curve equations above it appears that the surfaces corresponding will be of the fourth degree, thus proving Wangerin's assertion, and invalidating Haentzschel's criticisms.

HARVARD UNIVERSITY,
February, 1899.

PICARD'S ALGEBRAIC FUNCTIONS OF TWO VARIABLES.

Théorie des Fonctions algébriques de deux Variables indépendantes. Par ÉMILE PICARD, Membre de l'Institut, Professeur à l'Université de Paris, et GEORGES SIMART, Capitaine de Frégate, Répétiteur à l'École Polytechnique. Tome I, Paris, Gauthier-Villars et Fils, 1897. 8vo, vi+246 pp.

THE theory of functions of a single complex variable, the growth of which has been one of the striking features in the history of pure mathematics during this century, can, as it is well known, be developed from at least two tolerably distinct points of view. Cauchy and Riemann made extensive use both of methods of proof—such as integration—which may be conveniently called transcendental, and of geometrical reasoning; by such means they established not only results of a geometric or transcendental character but also others which, when once obtained, were naturally expressible in terms of pure algebra. Weierstrass and his school on the other hand have made scarcely any use of geometry, even for purposes of illustration, and have developed the subject from an almost purely algebraical standpoint, building it up systematically and logically from the most elementary notions. The former method has the interest which always attaches to investigations which connect two or more different branches of mathematics and use the methods of the one to solve problems in the other; and the geometric interpretations which continually present themselves are with most minds a valuable aid towards the clear comprehension of the theory. Partly owing to these reasons, partly owing to the well known difficulty of access to Weierstrass's ideas, most systematic treatises on the theory of functions which were published up to a few years ago expounded chiefly the ideas of Cauchy, Riemann and their followers. But, as the Weierstrassian methods have become more widely known, their severe simplicity, their unity, and their rigor have made many converts; and Weierstrass's dictum, that the theory of functions "must be built up on the basis of algebraic truths, and that it is not therefore the right way if conversely the 'transcendental' is employed to establish simple and fundamental algebraical propositions,"* is every year finding more general accept-

* Letter to Professor Schwarz, of October 3, 1875, Mathematische Werke, II, p. 235.

ance. Recent treatises illustrate this tendency. Dr. Forsyth's Theory of Functions and Messrs. Harkness and Morley's treatise on the same subject, each published some six years ago, were among the first books which gave an account of the subject from both points of view; and a like catholicity of treatment has shown itself in modern French and German treatises. More recently still Messrs. Harkness and Morley in their admirable Introduction to the Theory of Analytic Functions have employed Weierstrassian methods predominantly though not exclusively.

But the systematic methods which are possible and often desirable in expounding a subject which has been worked at with exceptional vigor and success for more than half a century, which has become a recognized subject of university teaching, and the boundaries between which and allied theories have been to some extent drawn, become inapplicable in dealing with a nascent subject, the difficulties of which have been only overcome here and there. As Weierstrass says in the letter already quoted: "It is evident that every path must be open to the investigator as long as he is still engaged in his inquiry."

The theory of functions of two or more independent variables, as at present known, belongs to this latter class of subjects, which are far too imperfect to admit of systematic treatment by a single method. The difficulties are such that any method which brings out results is welcome; even doubtful processes, which when carefully examined may be said rather to render their apparent conclusions plausible than to prove them, are of value, since results thus obtained are at any rate suggestive, and their accuracy can be tested at a later stage by other methods. In the same way it is permissible or even desirable to leave out of account exceptional cases in some particular part of the subject, reserving them for further treatment when some more results of a general character have been obtained, or to deal with special cases in the hope that they may throw light on a hitherto intractable general theory.

It is in this spirit that M. Picard and his collaborator, M. Simart, have written the present treatise; and in this spirit it should be read and appreciated. It is not, and from the nature of the case cannot be, a connected and systematic treatise dealing with functions of two variables, as Dr. Forsyth or Messrs. Harkness and Morley treat the corresponding case of one variable; its chapters may be described as a series of monographs dealing with those branches of the theory which have hitherto been most studied; but

unlike the original memoirs on which they are based they are written on a uniform plan and the connection between the different topics treated of is, within certain limits, made as evident as the nature of the case admits.

The methods employed correspond for the most part to those used by Cauchy and by Riemann for the study of functions of one variable; and the material of the book is to a large extent derived from memoirs by M. Picard himself and by M. Poincaré, while various results due to Noether, Clebsch, Betti, Cayley, Castelnuovo, Enriques and others have also been embodied. There is no reference to Weierstrass's theory of analytic functions of two or more variables, and no use is made of his methods.

It is well known that the theory of functions of one complex variable is intimately connected with geometry in two distinct ways. Some of the advantages of representing a complex variable $x(=\xi+i\xi')$ by a point in a plane were noticed more than a century ago; an important extension of the method was made by Riemann in introducing the surfaces now associated with his name for the treatment of many valued functions: the complex variable is thus represented by a point not confined to a single plane but moving either on a system of superposed planes connected in a particular way or on a surface, such as the familiar anchor ring, contained in space of three dimensions. The "branching" of the function receives an interpretation in certain of the geometrical properties of the corresponding Riemann surface. It is characteristic of this method that a correspondence is established between a single complex variable x and a real two dimensional locus, and that, though the method is admirably adapted for the study of what we may conveniently call the qualitative relations between the dependent variable or function (y) and the independent variable (x), the quantitative variations of y receive no interpretation.

An entirely different method consists in interpreting the two variables x, y as coordinates of a point in a plane; the functional relation between them gives a curve in the plane. But real points on the curve correspond only to real values of the variables; complex values of the variables only receive an interpretation when we generalize our geometrical conceptions so as to include imaginary points, and the actual diagrammatic representation fails. In this method, then, we establish a correspondence between our pair of variables and a one dimensional locus; and the quantitative variations of both are represented.

Thus if we are dealing with two variables x, y connected by the equation

$$y^2 = (1 - x^2)(1 - k^2x^2),$$

the first method employs the surface of an anchor ring, a double plane, or some other two dimensional locus, while the second interprets the equation as that of a plane quartic curve.

Some important differences arise when we pass from the case of one to that of two independent variables (x, y). The obvious extension of Riemann's geometric method requires for the representation of the four real variables involved in $x(=\xi + i\xi')$ and $y(=\eta + i\eta')$ a four dimensional locus,* and, just as in the case of one complex variable it may be convenient to work with a surface (like an anchor ring) contained in space of a higher number of dimensions, so this locus may be a curved "surface" in space of five dimensions. The second method of representation, however, only leads to an ordinary surface in space of three dimensions.

Thus, restricting ourselves to the study of *algebraic* functions of two variables, we have to deal on the one hand with the *Analysis situs* of hyperspace and on the other with the theory of ordinary algebraic surfaces.

The first two chapters of the book under review deal with some fundamental questions relating to the geometry of hyperspace and to integration in such space, these being necessary as an introduction to problems which are subsequently discussed.

If we restrict ourselves to ordinary space we have three kinds of integrals involving three variables, which present themselves commonly and are of geometrical importance, viz., line integrals of the type $\int(Pdx + Qdy + Rdz)$, surface integrals such as $\iint(Adydz + Bdzdx + Cdxdy)$ and volume integrals $\iiint Vdxdydz$. If we pass to space of n dimensions we have similarly m-tuple integrals in n variables, where m may have any integral value from 1 to n inclusive; the conditions of integrability of such integrals in the case of $m < n$, and the conditions that they should vanish when taken over a closed locus, relations between integrals of different orders, and allied questions are dealt with in a most suggestive though incomplete manner. The discussion of the *Analysis situs* of loci of any number of dimensions in space of a higher number of dimensions, which

* I use this word, though with some hesitation, as being on the whole the best English equivalent for *variété*.

follows, leads up to the definitions and to properties of certain numbers, which are, from one point of view, the generalization of the number known as the connectivity in the case of an ordinary Riemann surface. The connectivity of such a surface depends on the number of curves of an assigned character which can be drawn on the surface. But in the case of a higher number of dimensions it becomes necessary to consider in an analogous way not only curves (one dimensional loci) but also loci of 2, 3, $\cdots, n-1$ dimensions, where n is the number of dimensions of the locus considered. We thus have the conception of connectivities of different orders $p_1, p_2, \cdots, p_m, \cdots, p_{n-1}$, where p_m depends on the number of m-dimensional loci of a certain character which can be drawn on the surface. These numbers are called the *numbers of Betti and Riemann*, as they occur in a memoir by the former* and in a posthumous fragment published in the collected works of the latter.† It will be convenient to refer to them simply as Bettian numbers. It may be remarked in passing that it is perhaps a little unfortunate that this particular notation should have been chosen by MM. Picard and Simart, as when we revert to the case of $n = 2$, the number p_1 does not reduce, as the notation would suggest, to the familiar p of the ordinary theory, but is the connectivity $2p + 1$. These Bettian numbers are also shown, as was to be expected, to play an important part in the theory of integrals on the surface, the fundamental property being that if we are dealing with an n-dimensional locus E_n contained in space of a higher number of dimensions, then p_m is the number of independent periods obtained by integrating round closed circuits an m-tuple integral

$$\int \cdots \int \Sigma X_{1,2,\cdots,m} dx_1 dx_2 \cdots dx_m,$$

which satisfies the condition of integrability.

A remarkable property of Bettian numbers, which has, of course, no analogue in the theory of the ordinary Riemann surface, is that for a closed surface

$$p_m = p_{n-m},$$

a property established by M. Poincaré in an important memoir on the *Analysis situs* of hyperspace‡, but only proved by MM. Picard and Simart for the case of $m = 1$.

* *Annali di Matematica*, vol. 4 (1870–1871).

† Fragment aus der Analysis Situs, Werke (2d edition), pp. 479-482.

‡ *Jour. de l'École polytechnique*, series 2, vol. 1 (1895).

The following chapter deals chiefly with double integrals of functions of two variables, which are in general taken to be complex. The fundamental theorem is M. Poincaré's generalization* of Cauchy's well known theorem on the contour integration of a holomorphic function. The corresponding theorem when the function has singularities inside the locus over which the integral is taken is only established for the case of rational functions; and it is shown that corresponding to the residues which occur in Cauchy's theorem we now have periods of abelian integrals corresponding to the curve obtained by equating to zero the denominator of the subject of integration. Here also it may be permitted to make a trifling criticism of the definition of a residue of a function of two variables which M. Picard adopts after M. Poincaré, as the quantity thus defined corresponds, in the case of one variable, not to the residue which Cauchy's work has made classical, but to the same multiplied by $2\pi i$. The chapter concludes with a short discussion of simple integrals of total differentials in two complex variables, viz., of integrals of the type

$$\int (Pdx + Qdy)$$

where P and Q satisfy a condition of integrability, and are, moreover, restricted to being rational functions, but a fuller treatment of the subject is postponed to a later chapter.

The next chapter introduces the second of the two methods which we have referred to as giving a geometric interpretation of a function of two variables, viz., by an algebraic surface in ordinary space. Almost any problem of analytical solid geometry might of course be regarded as belonging to the theory of functions of two variables, but, as in the case of two dimensions, the characteristic ideas of the theory of functions are connected primarily with certain special parts of the theory of surfaces. In particular the theory of the singular lines and points of a surface, and the method of birational transformation play a leading part. The problem of the reduction of the higher singularities of plane curves, *e g.*, multiple points where two or more branches touch or have contact of a higher order with one another, to double points with distinct tangents is, as is well known, of great importance for various questions connected with the theory of functions of one variable. It has been extensively studied during the last quarter of a century and

* "Sur les résidus des integrales doubles," *Acta Math.*, vol. 9 (1887).

has been completely solved in a variety of ways by means of birational transformations. The corresponding problem in three dimensions is complicated by the existence of singularities of two distinct types, multiple lines and multiple points, while the latter may either be isolated or lie on the former. In reducing these singularities use is made of methods of projecting from space of higher dimensions which were, I believe, first used by Clifford* and have been extensively developed by modern Italian geometers. Just as, for example, a nodal plane cubic curve can be treated as the projection of a twisted cubic; so in general, a surface can be treated as the projection of a two dimensional locus in space of four or more dimensions with fewer singularities. The most general result of this kind established by MM. Picard and Simart is that any algebraical surface can be converted by projections into a "surface" in five dimensional space without singularities, and then, by further projections, into a surface in ordinary space, the only singularities of which are double lines with triple points on them. Thus isolated multiple points and lines with multiplicity higher than two can be removed. This projection establishes a one-one correspondence between the two loci, and when put into algebraical language is equivalent to birational transformation. Hence in problems in which the order of a surface is unimportant and we are dealing only with properties which are unaffected by birational transformation, it becomes unnecessary to consider singularities other than those just named.

The remainder of the chapter deals with the connectivity of algebraical surfaces. Since, as we have seen, the two complex variables (x, y) correspond to four real variables, there are three connectivities, p_1, p_2, p_3, to be considered, which, however, reduce to two in virtue of the equation $p_1 = p_3$. Purely geometrical methods of treatment, such as can be employed in the case of ordinary Riemann surfaces, soon become unmanageable owing to the difficulties of realizing hyperspace, and have to be supplemented by algebraical and transcendental processes; among the latter is the employment of a certain linear ordinary differential equations, from which the number p_1 for the surface can be obtained. A very remarkable difference between the numbers p_1, p_2, which at the same time emphasizes the difference between the cases of a plane curve and of a surface, is established. In the case of a plane curve C_n, represented by

* "On the classification of loci," *Phil. Trans.* (1878) and Math. Papers, pp. 305-331

$f(x, y) = 0$, it is well known that each additional singularity diminishes the genus (deficiency, genre, Geschlecht) of the curve, and consequently the connectivity of the corresponding Riemann surface; in particular the Riemann surface of a non-singular C_n is necessarily multiply connected if $n > 2$. The general effect of singularities on the connectivity of order $2(p_2)$ of an algebraical surface is of the same character, though not capable of such simple expression; the addition of a new singularity in general diminishes the number of possible two dimensional closed loci that can be drawn on the real four dimensional locus, which represents (in the sense already explained) the algebraic equation $f(x, y, z) = 0$. But the addition of a new singularity has, in general, an effect of the opposite kind on the connectivity of order one (p_1), as it tends to increase the possible number of closed curves that can be drawn; and in particular so far from a non-singular surface possessing the maximum number of such curves it is shown that it cannot possess any, and it is therefore simply connected.

The following chapter (V), on integrals of the first kind of total differentials, deals with one of the most interesting parts of the whole subject, the main outlines of it being due to M. Picard himself. In connection with a plane algebraic curve, $f(x,y) = 0$, we have abelian integrals $\int R(x,y)dx$ (where R is a rational function), which have been classified into the familiar three kinds. When we pass from a plane curve to a surface, $f(x,y,z) = 0$, we have to inquire in what direction we are to look for the equivalent of an abelian integral. One obvious generalization is to replace the single integral $\int R(x,y)dx$ by a double integral $\int\int R(x,y,z)dxdy$. Another and quite different generalization consists in taking an integral of a total differential, viz., $\int(Pdx + Qdy)$ where P and Q are rational functions of x,y,z, which satisfy a condition of integrability. These integrals like ordinary abelian integrals fall into three classes according to the nature of their infinities, and a special interest attaches to those of the first kind, which are finite at every point of the surface. Integrals of this kind were first introduced by M. Picard fifteen years ago*; and a very large part of what is now known about them is due to him. Though they do not seem to have been extensively studied, their importance both in geometry and differential equations may be seen by reference

* In a short paper in the *Comptes rendus*, December 1, 1884; soon followed by the memoir: "Sur les intégrales de différentielles totales algébriques de première espèce," *Liouville's Journal;* series 4, vol. 1 (1885).

to Humbert's memoir on hyperelliptic surfaces* and to Painlevé's Stockholm lectures on differential equations†.

A problem of capital importance which presents itself at the outset is that of ascertaining what surfaces admit of such integrals. This is by no means as simple as the corresponding problem in two dimensions. For whereas every plane curve of genus p admits of exactly p integrals of the first kind, so that only rational curves admit of no such integrals, the corresponding problem for surfaces is much more complicated and has by no means been completely solved. It is shown by a singularly beautiful piece of analysis (given in M. Picard's earliest paper on the subject), that the determination of surfaces, of a given order n, which admit of integrals of the first kind, depends on the integration of a linear partial differential equation. In homogeneous coordinates this equation can be expressed in the form

$$\theta_1 \frac{\partial f}{\partial x} + \theta_2 \frac{\partial f}{\partial y} + \theta_3 \frac{\partial f}{\partial z} + \theta_4 \frac{\partial f}{\partial t} = 0,$$

where the coefficients θ are quantics of order $n-3$, which satisfy the equation

$$\frac{\partial \theta_1}{\partial x} + \frac{\partial \theta_2}{\partial y} + \frac{\partial \theta_3}{\partial z} + \frac{\partial \theta_4}{\partial t} = 0.$$

For the case of $n = 4$ the integration of the equation presents no difficulties, except that the discussion of certain exceptional cases requires a little care. Two quartic surfaces are found which satisfy the required conditions, and it is stated without detailed proof that there are no others but these and their projections and certain cones. There is a trivial exception to this conclusion, which M. Picard appears to have overlooked, as a third surface can be found which, though it can easily be deduced by a limiting process from one of his surfaces, is not strictly included in it in the form in which it is given. In the case when $n > 4$, the integration of the differential equation presents formidable difficulties which do not appear to have been overcome. Several general results are, however, given which throw light on the existence or non-existence of surfaces of the kind considered. Thus in the case of a cone every abelian integral relative to a plane section (not passing

* *Liouville's Journal*, series 4, vol. 9 (1883).

† Leçons sur la théorie analytique des équations différentielles; Paris, 1897.

through the vertex) is obviously an integral of a total differential relative to the cone, so that if the section of the cone is a curve of genus p, we have p integrals of the first kind. As also a birational transformation converts an integral of the first kind of a total differential into an integral of the same character, surfaces which can be obtained from non-rational cones by a birational transformation will possess such integrals. On the other hand, a rational (unicursal) surface—the coordinates of a point on which are rational functions of two parameters—evidently possesses no such integrals, since the integral of a rational function is necessarily infinite for some values of the parameters. It follows that no surfaces of order 2 or 3 can possess any such integrals. Further, since one of our integrals must reduce to an abelian integral of the first kind for any algebraic curve on the surface, it follows that our integral must vanish identically along any rational (unicursal) curve on the surface; if, therefore the surface contains a family of rational curves, and there is an integral of the first kind, $\int(Pdx + Qdy)$, then $Pdx + Qdy = 0$ must be the differential equation of the family. MM. Picard and Simart also show that an integral of the first kind can only exist if the connectivity of the first order (p_1) is at least 3; and therefore cannot exist on a non-singular surface, for which it has already been shown that $p_1 = 1$, a result which is established independently by means of the differential equation.

Another subject of interest discussed in this chapter is the existence of functionally independent integrals of the first kind. An abelian integral associated with a plane curve is a function of one variable only, and therefore all such integrals are functions of one another; the ordinary theory deals only with *linear* independence. But integrals on a surface are in general functions of two independent variables, and we have therefore to consider cases of functional as well as of linear independence. Except in certain cases of hyperelliptic surfaces (which however are not discussed in the book) the results hitherto obtained are chiefly negative. The integrals of the first kind on a cone or on a surface obtained from it by a birational transformation as well as on any surface containing a family of rational curves, are obviously functions of one another; and it is further shown that there cannot be two functionally independent integrals on quartic surfaces or on certain classes of quintic surfaces.

Integrals of the second kind of total differentials, which form the chief part of the subject matter of chapter

VI, are in some respects both simpler and less interesting than those of the first kind. The characteristic of such integrals is that they are algebraically infinite at one or more points of the surface with which they are associated; consequently rational functions of the variables form an obvious though trivial class of such integrals, which it is in general unnecessary to consider further. The number of linearly independent integrals of the second kind, rational functions being ignored, is shown to be less by 1 than the number p_1 which expresses the connectivity of the first order. On the other hand the number of such integrals is shown to depend upon certain linear differential equations, and the study of these equations thus leads to a method of determining p_1, a number which it is not in general practicable to obtain by purely geometric methods.

A short discussion of integrals of the third kind concludes the chapter, but no results of special interest are obtained.

Double integrals which are always finite on an algebraic surface are also said to be of the first kind. The properties of these integrals and a number of important geometric questions with which they are intimately associated form the subject matter of the last two chapters. These integrals were first introduced by Noether,* and have subsequently been studied chiefly by M. Picard and, in connection with hyperelliptic surfaces, by Humbert. They form the most direct analogue of abelian integrals of the first kind and are closely connected with certain parts of the theory of birational transformation of surfaces. In the theory of plane curves there is only one known number p which is invariant for such transformation; it can be defined in at least four ways, viz., (1) in terms of the connectivity of the associated Riemann surface by means of the equation $2p+1=N$, (2) by means of the formula $p=\frac{1}{2}(n-1)(n-2)-2\delta$, connecting it with the order of the curve and the number of double points (or equivalent higher singularities), (3) as equal to the number of linearly independent integrals of the first kind, (4) as equal to the number of adjoint polynomials of order $n-3$. We have already seen that when we extend to surfaces the ideas involved in the first of these definitions we meet with two distinct Bettian numbers p_1, p_2. Corresponding to the third and fourth definitions we have for a surface of order

* "Zur Theorie des eindeutigen Entsprechens algebraischer Gebilde von beliebig vielen Dimensionen," *Math. Annalen*, vol. 2 (1870).

n an invariant number* (p_g), called the geometric genus (*genre géometrique*) which is equal to the number of linearly independent double integrals of the first kind, as well as to the number of linearly independent adjoint surfaces of order $n-4$, the definition of adjoint being appropriately modified. Corresponding to the second definition we have a number (p_a), also invariant, called the numerical genus (Cayley's deficiency), which in the case when the surface has only singularities of the simplest kinds is defined by the formula

$$p_a = \tfrac{1}{6}(n-1)(n-2)(n-3) - \Sigma\tfrac{1}{6}i(i-1)(i-2)$$
$$-(n-4)d + 2t + \pi - 1$$

where the second term on the right is due to isolated i-tuple points, and the subsequent terms to a double curve of order d and genus π with t triple points on it. Analogy with the plane case might lead us to suppose that we should always have $p_g = p_a$; but this proves not to be the case. By the definition p_g is positive or zero; but it was pointed out by Cayley in 1871† that p_a might be negative, and that in particular for a ruled surface p_a was equal to the genus of a plane section taken negatively. For example, in the case of a quartic cone without double lines, the formula just quoted gives

$$p_a = 1 - 4 = -3.$$

Subsequent investigation has shown that, though in a certain large sense the two numbers are generally equal, there are important classes of surfaces for which $p_g > p_a$, while the inequality is never reversed. A slight sketch is given of some of the leading geometrical properties with which these numbers are associated, and of their connection with double integrals of the first kind. The theory of double integrals is also applied to prove the invariance for birational transformation of still another "genus" of surfaces (*Curvengeschlecht, second genre*), which can be defined as the genus (in the ordinary sense) of the variable curves of intersection of the original surface and the adjoint surfaces of order $n-4$; with this are again associated other numbers showing the fundamental property of invariance.

* First introduced by Clebsch, "Sur les surfaces algébriques," *Comptes rendus*, vol. 67 (December, 1868).

† "On the deficiency of certain surfaces," *Math. Annalen*, vol. 3; Coll. Math. Papers, vol. 8, no. 524.

The final chapter is chiefly geometrical, dealing partly with some of the questions already referred to, and especially with certain topics in which the theory of curves in space is associated with that of surfaces. It is necessarily fragmentary in character, and may, perhaps, be regarded in some sense as an introduction to the more purely geometric treatment of surfaces which M. Picard promises us in his second volume.†

It is difficult to give in a few words a critical verdict on a book which embraces as many subjects as the volume under review. It is undoubtedly rather fragmentary; and the reader who goes straight through it, and does not merely pick out the chapters which deal with topics of which he has made the acquaintance elsewhere, may be to some extent inclined to complain that the connection between different parts of the book is not made more evident. One would find it rather a relief—to take one illustration—if a clearer idea were given of the relation between the Bettian numbers which are dealt with in the earlier chapters, and the various "genera" which figure prominently towards the end, all of them in some sense analogues of the familiar and unique p of two dimensional geometry. And again a reader whose tastes lie in the direction of thoroughness in investigations may have some cause for complaint on account of the frequency with which the authors refuse to discuss any but the most straightforward cases of the various problems; it is difficult to estimate the value of a theorem which only professes to be "generally" true.

But this scrappiness, which I believe very largely to be due to the present state of our knowledge of functions of two variables, has great counter-balancing advantages. There is hardly a chapter in the book which does not suggest to any intelligent reader a variety of problems which cry for solution. Some of these are probably worthy the attention and require the skill of M. Picard's equals, while to others any competent analyst or geometer might devote some attention with reasonable expectation of obtaining results of interest. Above all things the book is supremely interesting; for my own part, at least, I can recall no book that

* Introduced by Noether in his paper: "Zur Theorie des eindeutigen Entsprechens algebraischer Gebilde," *Math. Annalen*, vol. 8 (1875).

† The reader to whom the general geometrical questions touched on in this and the preceding chapters are not familiar may be referred to a most fascinating article by Castelnuovo and Enriques, "Sur quelques récents résultats dans la théorie des surfaces algébriques" (*Math. Annalen*, vol. 48 (1897)).

I have read with such pleasure since the days when I first met with Dr. Salmon's incomparable treatise on conic sections.

ARTHUR BERRY.

KING'S COLLEGE, CAMBRIDGE

NOTE ON PAGE'S ORDINARY DIFFERENTIAL EQUATIONS.

AN interesting review of this elementary text book was given by Professor Lovett in the BULLETIN, April, 1898. As the suggestions offered in the review cited are mainly of a general nature and appeal especially to those teachers familiar with the larger works of Lie, and hence able to make the desirable amplifications, it would seem worth while to address to the average reader or teacher of this text a few critical remarks of detailed character. Since my first acquaintance with Lie's groups and theories of integration, I have had the desire to introduce a class of mature students to the theory of ordinary and partial differential equations through the medium of continuous groups. Having used* the text by Page, I am more than ever convinced that the proper method (and one that will come more and more into vogue) of attacking differential equations is that which employs the powerful machinery—so simple when once mastered—set up and perfected by the illustrious Lie.

Being in full sympathy with the aims of the text, I was glad to find that, on the whole, the task had been well executed. I trust that in a second edition all objections that prove to be well grounded will be eradicated and that the errata, too numerous for an elementary text, will be corrected.

There is a curious mistake on p. 6, where the tangents to every integral curve of an ordinary differential equation are said to pass through the origin! This is indeed the case for the only example given in the paragraph concerned. The answer to Ex. (19), p. 9, should be

* During a year's graduate course in continuous groups, we devoted two months to the reading of Page's text, finding it a very practical supplement to a course of lectures on the general theory.

$$(x^2 + y^2)y'' - xy'^3 + yy'^2 - xy' + y = 0.$$

It would be well to establish the *vice versa* of p. 17, perhaps by use of formulæ (3), p. 11.

That the once extended (or m times extended) transformations of a one parameter group form a group seems to me unnecessary of proof. Indeed, a given transformation in x and y gives rise to definite transformations of the derivatives $y', y'', \cdots$ and it is entirely a matter of choice or convenience whether or not we will include in the *symbol* of the transformation not only the increments of x and y, but also the increments of $y', y'', \cdots$ If a formal proof be demanded, that proof should be as clear as possible. Instead of abridging slightly Lie's formal proof,* Page in his proof, p. 57, might well have amplified the derivation of (8) from (5), (6) and (7).

Examples (10) of p. 60 and (12) of p. 61 are quite faulty. The student is required to apply Art. 40, which is concerned with a family of ∞^1 curves whose equations may, therefore, be solved for the one arbitrary constant entering it. But the examples cited are concerned with families of ∞^2 curves. This difficulty may be obviated in Ex. (10) by considering separately the sub-families of ∞^1 conics in each of which the parameter b has any *particular* value. A *second* error in (10) lies in the incompatibility of the relations

$$\omega \equiv \frac{x^2}{a^2} \pm \frac{y^2}{b^2}, \quad U(\omega) \equiv x\frac{\partial \omega}{\partial x} \equiv \frac{2x^2}{a^2} = \Omega(\omega).$$

It is a pity that the proper historical setting was not given to the developments on pp. 69–71. In fact the investigation is identical with Lie's *first* (1869) method of integrating an ordinary differential equation of the first order admitting a known one parameter group. This earlier theorem proves that an ordinary differential equation admitting a known one parameter group *whose path curves are known* can be integrated by two quadratures. It is true that Page, carrying out the suggestion of Lie (l. c., p. 117), uses this investigation for the discussion of the problem to set up *all* differential equations of the first order which admit a given one parameter group. Although the developments given by Page really prove the integration theorem, no mention is made of the latter. He evidently prefers the later (1874) method given on p. 75.

* Lie-Scheffers, Vorlesungen über Differentialgleichungen, p. 267.

It is to be regretted that Page did not devote a little space to the consideration of ordinary differential equations admitting two essentially distinct infinitesimal transformations, especially as there results the simple and very important theorem that the quotient of the two resulting integrating factors is an integral of the given differential equation. The proof requires but a few lines (Lie-Scheffers, p. 124). The simple relation holding between two such infinitesimal transformations is readily deduced in a number of ways (Lie-Scheffers, pp. 125–132).

With reference to the table, pp. 96–97, I wish to call attention to the fact that one of the members of my course, Mr. Hathaway, has set up by general methods two very general types of infinitesimal transformations with the corresponding invariant differential equations. Most of the types given by Page are special cases of the following infinitesimal transformation involving three arbitrary functions:

$$-F(x)\frac{\partial f}{\partial x}+\{\varphi(x)F(x)y+\theta(x)F(x)\}\frac{\partial f}{\partial y},$$

leaving invariant the differential equation

$$\mu y'=\frac{1}{F}\omega(\mu y+\nu)-(\mu' y+\nu')$$

where ω denotes an arbitrary function of its argument and where

$$\mu\equiv e^{\int\phi(x)dx},\quad \nu\equiv\int\mu\theta dx.$$

In this way Mr. Hathaway noted the error in the type (12) of p. 97, where xy' should read y'/x. Perhaps the error crept in by analogy to type (10).

We should have welcomed in Chapter V some examples of differential equations representing families of isothermal curves in addition to the two examples, viz., (1) p. 106 and (8) p. 10*i*, taken from Lie-Scheffers.

The discussion in § 82 of differential equations of *degree* higher than the first might well be revised. The force of the word "rational," put in italics, is not clear; nor the reason for writing an integral in the form $y-\varphi(x, y, c)$ instead of the customary form $\varphi(x, y)-c$.

In the exposition in Chapter VII of Boole's treatment of Riccati's differential equations, Page uses as ultimate forms certain integrable differential equations in which the variables are not separate, whereas he might with equal ease

have given forms with the variables separate. For example, why ask the reader to put the simple equation (3), at the bottom of page 120, into the cumbersome form (4) in order to integrate it? The remark on p. 120 that equation (2) is "much more easy to discuss" than equation (3) leads me to say that, in common with many others, I prefer to discuss the latter, but in the form to which it is easily reduced

$$(E) \qquad \frac{dy}{dx} + \tfrac{1}{2}y^2 = x^m.$$

This is Euler's special case of the reduced form of the *general* Ricatti's equation, the latter reduced form having an arbitrary function $\psi(x)$ in place of x^m in its right member. The cases in which (E) is integrable are found by using two simple types of substitutions, each transforming both the independent and the dependent variables. The discussion of the asymptotic case $m = -2$ would be of interest to the reader.

I do not understand the expression on p. 129 "loci composed of multiple points, cusps, etc." In order to speak of "the condition" on page 134, an inverse theorem would have to be established. The error of notation of using X_{n-1} for X_n occurs on pages 171, 175, 176 and 179. On p. 180, line 7, "the left member of" should be inserted after "in." At the bottom of p. 185, it is *proven* that the resulting equation is free from y, not that it is linear in v, a result sufficiently evident however.

A revision of §149 would be welcomed. Given X, Y, Z, the functions λ, μ, ν can *always* be found such that

$$(I) \qquad \lambda X + \mu Y + \nu Z = 0;$$

indeed, if $X \neq 0$ for example, we may choose μ and ν arbitrary and solve for λ. But the resulting equation (3) would in general be sufficiently difficult to integrate. That an integral (3) is "obviously" an integral of (1) had to be proven to my class of able graduate men. We may give a simple proof as follows: We are given that

$$\frac{\partial \Omega}{\partial x}dx + \frac{\partial \Omega}{\partial y}dy + \frac{\partial \Omega}{\partial z}dz = 0$$

is a consequence of (3). Hence must

$$\lambda : \mu : \nu \equiv \frac{\partial \Omega}{\partial x} : \frac{\partial \Omega}{\partial y} : \frac{\partial \Omega}{\partial z}.$$

Then (I) shows that Ω is a solution of the partial differential equation

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z} = 0,$$

equivalent to the simultaneous system (1).

Evident misprints occur on p. 145, l. 7, p. 157, p. 182. It adds clearness to use $y \cot nx$ instead of $\cot nxy$ used p. 188.

A final remark is that it seems preferable to teach a general *method of procedure* for solving differential equations using freely transformations of the independent and dependent variables, rather that the application of a general formula. For example, the integration of the general linear differential equation of the first order is performed by a simple method, but by a complicated formula.

L. E. DICKSON.

UNIVERSITY OF CALIFORNIA,
March 23, 1899.

TANNERY'S ARITHMETIC.

Leçons d'Arithmétique théorique et pratique. By JULES TANNERY. Paris, Colin et Cie, 1894. viii+509 pp.

THE present volume from the pen of the distinguished director of scientific studies at the École Normale Supérieure in Paris is the first work on arithmetic we have seen which while intended entirely for secondary instruction is written in accordance with the new ideas regarding the number concept and the need of rigor. It is thus a pioneer, perhaps even the inaugurator, of a revolution in secondary instruction in mathematics and as such will receive praise or censure according as the person in question is thoroughly awake to the crying necessity of reform in secondary mathematical instruction, or is not.

For fifty years or more slow changes have been taking place in the mathematical world. Their cumulative effect has completely transformed the aspect of mathematics from its bottommost foundations to the summit. Such mathematicians as Gauss, Cauchy, and Abel found the great structure of mathematics almost without foundation. Here is an extract of a letter of Abel to Hansteen, dated 1826: "Je

consacrerai toutes mes forces à répandre de la lumière sur l'immense obscurité qui règne aujourd'hui dans l'analyse. Elle est tellement dépourvue de tout plan et de tout système qu'on s'étonne seulement qu'il y ait tant de gens qui s'y livrent—et ce qui est pis, elle manque absolument de rigueur." In a letter to Holmboe he writes: "Enfin mes yeux sont dessillés d'une maniére frappante, car à l'exception des cas les plus simples, par example les séries géométriques, il ne se trouve dans les mathématiques presque aucune série infinie dont la somme soit determinée d'une manière rigoureuse, c'est à dire que la partie la plus essentielle des mathématiques est sans fondement." Under the leadership of Gauss, Cauchy, and Abel a great reform set in. The most important factor in this process was the number concept. In order to gain a basis on which the magnificent theories of our predecessors could be securely based it was found in last analysis that the difficulties were purely arithmetical. To overcome these difficulties it was necessary to create a new theory of number, quantity, and magnitude. Today we have not only precise ideas on continuous number but also a satisfactory theory of extensive number or *polyplets* of which complex numbers are the simplest type, as well as a theory of magnitude in general. These great achievements have as yet had no influence on text books of secondary grade. The arithmetics, algebras, trigonometries, and works on calculus, with a few conspicuous exceptions, are no whit better in this respect than they were a generation ago. The absolutely false, or lamentably imperfect and inadequate ideas of the analysts of the last century we find repeated *ad nauseam* in nearly every new text book on these subjects.

In our opinion the time has come to sound an alarm, to awaken our educators to the fact that it is not just or wise to teach our youths notions which, if not altogether obsolete, have been so profoundly transformed as to be hardly recognizable. It is not the circle of ideas of the last century that we should instill in their minds but those of this end of the nineteenth century. What we wish to see is a series of text books which begins at the beginning, viz., arithmetic, conducts our youth through our college mathematics, and which is written with due attention to the present state of mathematics and the present demands made in regard to rigor.

To have advocated such a course before the appearance of Tannery's *Leçons* would no doubt have seemed ridiculous to many. We should have been overwhelmed by *à priori* argu-

ments drawn from the science of pedagogics against its possibility. Venerable colleagues, grown white in the service of instruction, would have given us up as dreamers of impracticable things. To all these and to those who are truly eager for progress, who wish to see secondary mathematical instruction in this country abreast of the times we say: read the *Leçons* and then reflect.

The most difficult part of our program is the first volume in the series, the arithmetic. How to present our modern notions on this subject clearly and rigorously has seemed to us, who have long cherished these ideas, a task so arduous that few could hope to attain even fair success. And yet a correctly yet simply written arithmetic is the key to the situation, a *sine qua non*. We consider it a fortunate circumstance for the cause of a better secondary instruction in mathematics that the first attempt in this direction fell to no less able hands than those of M. Tannery. We confess gladly that his *Leçons* have been a revelation to us. He has solved so many of the difficulties which beset his path with such admirable tact that we believe future authors will be obliged to follow largely the path he has opened. At any rate the *Leçons d'Arithmétique* are for us an ocular demonstration that a series of text books such as we have contemplated is not only possible but desirable. We believe that other mathematicians will now be encouraged by M. Tannery's success and that the time is not far distant when our mathematical text books of secondary grade will be as excellent and as up to date as those written today in connection with university instruction.

We extend our sincere congratulations to M. Tannery for his bold and successful innovation. If we have not reviewed his book in detail it is because we have believed that we could employ the limited space at our disposal to best advantage in signalizing its nature and its real importance.

JAMES PIERPONT.

YALE UNIVERSITY,
May, 1899.

NOTES.

THE Toulouse Academy of Sciences announces for the year 1901 the following subject for its mathematical prize of five hundred francs:

"The investigation of the families of surfaces possessing the property that all their orthogonal trajectories are plane curves, from one of the following points of view: 1° In order that all the surfaces defined in rectangular coördinates by the equation $\rho = f(x, y, z)$, where ρ is a parameter varying from one surface of the family to another, may have plane orthogonal trajectories, it is necessary that f should satisfy a partial differential equation of the third order. The study of this equation is proposed. 2° The perimorphic method may be employed, reference being made to the 'Mémoire sur la théorie générale des surfaces courbes' of Ribaucour (*Liouville's Journal*, ser. 4, vol. 7), particularly to chapter 13, entitled 'Recherches des trajectoires orthogonales planes des surfaces.'"

Manuscripts may be written in French or Latin and must be deposited with the Secretary of the Academy before January 1, 1901. The usual conditions as to anonymity prevail.

AMONG the prize subjects announced by the Belgian Academy of Sciences for the year 1900 are two of mathematical character: 1° The history and theory of the variation of latitude. 2° The algebra and geometry of n-linear forms where $n > 3$. The prizes are gold medals of the value of six hundred francs each. Manuscripts may be written in French or Dutch, and must be sent to the Secretary of the Academy before August 1, 1900.

The second volume of the second edition of Professor H. WEBER'S Lehrbuch der Algebra has just appeared from the press of Vieweg und Sohn.

THE seventy-first meeting of the German association of naturalists and physicians will be held at Munich, September 18–23, 1899.

THE UNIVERSITY OF CHICAGO. During the four quarters (*su, a, w, sp*) of the year July, 1899–June, 1900, the following advanced mathematical courses (four or five hours weekly) will be offered:—By Professor MOORE: Algebraic numbers (introductory course, followed by a seminar) (*w*,

sp); Advanced algebra (*a*, *w*); Advanced integral calculus I (*a*); Functions of a complex variable I (*sp*).—By Professor BOLZA: Hyperelliptic functions (*w*); Seminar on hyperelliptic functions (*w*); Theory of groups (*su*); Functions of a complex variable I (*su*).—By Professor HATHAWAY (of the Rose Polytechnic Institute): Quaternions (*su*).—By Associate Professor MASCHKE: Linear differential equations (*sp*); Functions of a complex variable II (*a*); Theory of invariants (*a*); Twisted curves and surfaces (*w*, *sp*); Advanced integral calculus II (*w*).—By Assistant Professor YOUNG: Theory of numbers (*a*), Pedagogy of mathematics ($\frac{1}{2}$ *su*), Determinants ($\frac{1}{2}$ *su*).—By Dr. BOYD: Twisted curves and surfaces (*su*). Solid analytics (*sp*).—By Dr. SLAUGHT: Differential equations (*su*, *sp*).

The Mathematical Club, with fortnightly meetings, is under the direction of the departmental faculty.

COLUMBIA UNIVERSITY. During the academic year 1899–1900 the following advanced courses will be given by the department of mathematics, each course occupying three hours a week throughout the year:—By Professor FISKE: Advanced calculus: Theory of functions of a complex variable.—By Professor COLE: Theory of invariants; Theory of groups.—By Dr. MACLAY: Analytical theory of curves of double curvature and curved surfaces.—By Mr. KEYSER: Differential equations.

HARVARD UNIVERSITY. During the year 1899–1900 Professor OSGOOD will be absent in Europe on leave. During the first half of the academic year Professor PIERPONT, of Yale University, will give a course on Algebraic Numbers. The other advanced courses offered are:—By Professor J. M. PEIRCE: Quaternions (first course); Algebraic plane curves; and *either* Quaternions (second course) *or* Linear associative algebra.†—By Professor ASAPH HALL (U. S. N.): Theory of planetary motions.—By Professor BYERLY: Dynamics of a rigid body.†—By Professors BYERLY and B. O. PEIRCE: Fourier's series, spherical harmonics, potential function.—By Professor B. O. PEIRCE: Hydromechanics.—By Professor BÔCHER: Infinite series and products;† Higher algebra;† Linear differential equations.—By Dr. BOUTON: Theory of equations and invariants;† Theory of functions (first course).—By Mr. WHITTEMORE: Modern geometry; Differential and integral calculus (second course); Calculus of variations.†

These courses will each involve three lectures a week throughout the year, except those marked †, which involve

about half this number of lectures. The following courses of reading and research are also offered:—By Professor HALL: Selected topics in celestial mechanics.—By Professor BÔCHER: Linear differential equations.—By Dr. BOUTON: Continuous groups.

A mathematical conference will meet twice a month.

YALE UNIVERSITY. The following graduate courses in mathematics are announced for the session of 1899–1900: By Professor CLARK: Determinants and theory of equations, two hours, first term; Differential equations, two hours, second term.—By Professor GIBBS: Vector analysis, three hours; Electricity and magnetism, one hour; Thermodynamics and properties of matter, two hours.—By Professor BEEBE: Computation of orbits, two hours.—By Professor PIERPONT: Differential equations, three hours; Elliptic functions, two hours; Algebraic numbers and functions, two hours.—By Professor SMITH: Theory of transformations in space, two hours; Algebraic curves and surfaces, two hours.—By Dr. WESTLUND: Advanced calculus, three hours.—By Dr. STRONG: Higher algebra, three hours, first term.—By Mr. HAWKES: Modern analytic geometry, three hours, second term.—By Dr. STARKWEATHER: Mechanics, two hours.

THE number of the *Bolletino di bibliografia e storia delle scienze matematiche* for April, May, and June contains an appreciative account by Professor Segre of SOPHUS LIE and his work.

THE initial number of the current volume of the *American Journal of Mathematics* is accompanied by a photo-engraving of the portrait of Professor SIMON NEWCOMB, which was recently presented to Johns Hopkins University. Portraits of Sylvester, Cayley, Hermite, Klein, Poincaré, Fuchs, Lie, Halphen, and Darboux have appeared in previous volumes.

Professor THOMAS CRAIG, of Johns Hopkins University, has withdrawn from the editorship of the *American Journal of Mathematics* after seventeen years' service as associate editor and editor. The *Journal* is now in charge of Professor SIMON NEWCOMB.

LORD RAYLEIGH and Professor G. H. DARWIN have been elected honorary members of the New York Academy of Sciences.

PROFESSOR CHARLOTTE ANGAS SCOTT, of Bryn Mawr College, has been elected an honorary member of the Amsterdam Mathematical Society.

AT the University of Göttingen Dr. GEORG BOHLMANN has been promoted to a professorship of mathematics, and Drs. J. SOMMER and E. ZERMELO have been appointed docents in mathematics.

DR. FLORIAN CAJORI, formerly professor of physics at Colorado College, has been made professor of mathematics in the same institution, and Dr. S. J. BARNETT has been promoted to the professorship of physics.

DR. L. A. BAUER has resigned his position as assistant professor of mathematics and mathematical physics at the University of Cincinnati, having been appointed chief of the new division of terrestrial magnetism in the U. S. Coast Survey. Dr. F. H. SAFFORD, instructor in mathematics at Harvard University, has been appointed to the position formerly held by Dr. Bauer.

DR. F. C. FERRY, of Clark University, has been elected assistant professor of mathematics in Williams College.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BERGER (A.). Undersökningar öfver nagra aritmetiske Funktioner. Stockholm, 1898. 8vo. 41 pp. M. 1.50

—— Om de konvexa Polyedrarne. Stockholm, 1898. 8vo. 26 pp. M. 1.20

BITELLI (G.). L'insegnamento delle misure lineari superficiali volumetriche. Bologna, 1899. 12mo. 50 pp. Fr. 1.50

BRUN (F. DE). Einige neue Formeln der Theorie der elliptischen Functionen. Theil II. Stockholm, 1898. 8vo. 9 pp.

BURKHARDT (H.). Funktionentheoretische Vorlesungen. Zweiter (Schluss-) Theil. Elliptische Funktionen. Leipzig, Veit & Co., 1899. 8vo. [illegible] and 373 pp. M. 10.00

ENCYKLOPÄDIE der mathematischen Wissenschaften. Erster Theil, Band I., zweites Heft. Leipzig, Teubner, 1899. M. 3.40

ESCHERICH (G. v.). Die zweite Variation der einfachen Integrale. Mittheilung 1-3. Vienna, 1899. 8vo. 60, 60, 48 pp. M. 3.10

FRANCESCHI (G.). I più importanti teoremi di aritmetica ragionata. Naples, 1899. 12mo. 140 pp. Fr. 2.25

GRÖNWALL (H.). Sur les fonctions qui ne satisfont à aucune équation différéntielle algébrique. Stockholm, 1898. 8vo. 9 pp. M. 0.80

LIPSCHITZ (R.). Bemerkungen über die Differentiale von symbolischen Ausdrücken. Berlin, 1899. 8vo. 15 pp. M. .50

MEBIUS (C. A.). Ueber die Ableitung der Maxwell'schen Differentialgleichungen aus dem Hamilton'schen Principe. Stockholm, 1898. 8vo. 7 pp.

MITTAG-LEFFLER (G.). Om den analytiska Främställningen af en allmän monogen Funktion. III. Stockholm, 1898. 8vo. 11 pp.

NEPPI-MODONA (A.) E VANNINI (T.). Questione e formole di geometria analitica (ad una e due dimensioni). Secunda edizione. Palermo, 1899. 8vo. Fr. 4.00

PIETSCH (C.). Katechismus der Raumberechnung. Anleitung zur Grössenbestimmung von Flächen und Körpern jeder Art. Vierte, verbesserte Auflage. Leipzig, 1899. 8vo. 8 and 124 pp. Cloth. M. 1.80

PUZYNA (J.). Theory of analytic functions. Volume I. Lemberg, 1898. 8vo. 18 and 549 pp. (Polish). M. 14.00

VANNINI (T.). See NEPPI-MODONA (A.).

II. ELEMENTARY MATHEMATICS.

BAGGI (V.). See BOCCARDO (E. C.).

BOCCARDO (E. C.) e BAGGI (V.). Trattato elementare completo di geometria pratica. Torino, 1899. 8vo.

COMPENDIO di arithmética; por los Hermanos de las escuelas cristianas. Para uso de las clases infimas. Correspondiente á los cursos preparatorio y elementale. Duodecima edicion. Tours, Mame, 1899. 18mo. 126 pp.

JACKSON (J.). Practical arithmetic on an entirely new method for schools, examination candidates, etc. London, Low, 1899. 8vo. 460 pp. 4s.

LACHLAN (R.). Elements of Euclid. Book I. Revised edition. London, Arnold, 1899. 8vo. 152 pp. 1s.

MACDONALD (R. F.). A school arithmetic. London, Macmillan, 1899. 8vo. 272 pp. 2s. 6d.

NESBITT (H. A.). See SONNENSCHEIN (A.).

O'SULLIVAN (D.). Principles of arithmetic: a complete text-book for use of teachers and advanced pupils. 15th edition, revised and enlarged. London, Simpkin, 1899. 8vo. 432 pp. 3s. 6d.

—— Practice of Arithmetic: a companion volume to the principles of arithmetic. Part I. 38th edition. London, Simpkin, 1898. 12mo. 248 pp. 2s.

—— Key to practice of arithmetic. Parts I. and II. 5th edition. London, Simpkin, 1899. 12mo. 92 pp. 3s.

PLESSNER (P.). Lehrbuch der Stereometrie oder Körperlehre nebst Aufgabensammlung und 4-stelligen Rechentafeln. Köln. 1899. 8vo. 8 and 98 pp. M. 1.60

PRESSLAND (A. J.) and TWEEDIE (C.). Elementary trigonometry. London, Simpkin, 1899. 8vo. 160 pp. 2s.

SANDERSON (F. W.). Geometry for young beginners: an introduction to theoretical and practical geometry. New York, The Macmillan Co., 1899. 16mo. 9 and 132 pp. $ .35

SCHAPOSCHNIKOFF (N.) and WALZOFF (N.). Collection of algebraic examples. Part II. 6th edition. Moscow, 1898. 8vo. 190 pp. (Russian). M. 2.50

SONNENSCHEIN (A.) and NESBITT (H. A.). New science and art of arithmetic for schools. London, Sonnenschein, 1899. 8vo. 516 pp. 4s. 6d.

TREGAN (T. H.). Elementary and intermediate algebra. 10th edition. 8vo. 596 pp. 6s.

TWEEDIE (C.). See PRESSLAND (A. J.).

VACCHA (D. H.). Key to algebraical factors and their application to various processes in algebra (for beginners). 3d edition, revised and enlarged. New York, Longmans, 1898. 12mo. 6 and 187 pp. $.75

VEGA-VERDUGO (J. M.). Algebra, escrita con sujecion al programa vigente para los examens de ingresso en la escuela naval flotante. Madrid, 1898. 4to. 288 pp. Fr. 13.00

WALZOFF (N.). See SCHAPOSCHNIKOFF (N.).

III. APPLIED MATHEMATICS.

BARBARO (L.). Sull' equilbrio d'una sfera e d'un involucro sferico solidi, elastici ed isotropi sotto l'azione di forze di massa provenienti da un potenziale di secondo grado. Messina, 1899. 8vo. 31 pp. Fr. 2.50

BIK (A.). Elements of geodesy. Volume III. 2d edition. Moscow, 1898. 8vo. 384 pp. (Russian). M. 12.00

CAMPBELL (W. W.). The elements of practical astronomy. 2d edition. Revised and enlarged. New York, The Macmillan Co., 1899. 12 and 164 pp. $2.00

DEVILLE (E.). Photographic surveying, including the elements of descriptive geometry and perspective (Ottawa Gov. Series) London, Low. 12mo. 250 pp. 10s. 6d.

HURST (J. T.). Handbook of formulas, tables and memoranda for architectural surveyors and others engaged in building. 15th edition. New York, Spon & Chamberlain, 1899. 32mo. 512 pp. Leather. $2.00

MARBE (K.). Naturphilosophische Untersuchungen zur Wahrscheinlichkeitslehre. (Grundzüge der Wahrscheinlichkeitsrechnung; Untersuchungen über öffentliche Glücksspiele, etc.). Leipzig, 1899. 8vo. M. 1.20

MEYER (O. E.). Die kinetische Theorie der Gase. In elementarer Darstellung mit mathematischen Zusätzen. Zweite Auflage. 2. Hälfte. Breslau, Maruschke and Berendt. 8vo. 16 and pp. 145–352 and 65–128. M. 7.00

PERCHOT (J.). See TISSERAND (F.).

POINCARÉ (H.). See TISSERAND (F.).

REULEAUX (F.). The constructor: a handbook of machine design; authorized translation, complete and unabridged from the 4th enlarged German edition, by H. H. Suplee. Milwaukee, C. N. Caspar, 1899. $7.50

RUFFINI (F. P.). Ricerche intorno ai momenti d'inerzia di un sistema di punti privo di baricentro. Bologna, 1899. 8vo. 26 pp. Fr. 2.50

SCHULTZ-STEINHEIL (C. A.). Eine Methode, den Jupitersradius zu bestimmen. Stockholm, 1898. 8vo. 11 pp.

STRÖMGREN (E.). Ueber Kometenbahn-Excentricitäten. Theil I. Stockholm, 1898. 8vo. 30 pp. M. 2.00

SUPLEE (H. H.). See REULEAUX (F.).

TISSERAND (F.). Leçons sur la détermination des orbites, professées à la Faculté des Sciences de Paris. Rédigées et développées pour les calculs numériques, par J. Perchot. Avec une préface de H. Poincaré. Paris, Gauthier-Villars, 1899. 4to. 124 pp. Fr. 3.50

WADDELL (J.). The arithmetic of chemistry: being a simple treatment of chemical calculations. New York, The Macmillan Co., 1899. 12mo. 8 and 133 pp. $.90